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On the Seismic Disturbance Rejection of Structures*

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Abstract. Using results of disturbance rejection in optimal control and computational mechanics' techniques, a new approach for the design of robust structural control systems for aseismic design applications is presented. A shear-type frame structure is used to outline the proposed methodology.

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1. Introduction

In this paper the seismic disturbance rejection problem of civil engineering structures via state feedback control is investigated. Civil engineering structures modeled by matrix structural analysis techniques are examined. It is assumed that the only disturbance inputs are the seismic acceleration of the ground. The control forces inputs are placed on nodes of the floor over the foundation (control story). Goal of the control problem is the reduction of the displacements of the whole system. It is found that for this structural control system the seismic disturbance rejection problem always has a solution. The general analytical expression of the controller is also studied.

Optimal control problems of structures have been studied in several works in the recent years. Practical applications at multistory buildings and bridges have also been constructed. Studying the theory of automatic control one recognizes that the disturbance rejection problem is one of the most serious ones, with enormous practical applications. Its aim is the elimination of the influence of the disturbances in the system output. An earthquake, blast or wind loading is certainly a disturbance for a civil engineering structure. In general, the disturbance rejection problem for a civil engineering structure does not have a solution [7]. Specific control configurations, for instance for the shear-type frame used as an example here, make this problem solvable. The results for left invertible systems given in [4] are applied in

* This paper is dedicated to the memory of Professor P.D. Panagiotopoulos

this paper. More details on the notation used here, as well as useful results on the subject and H^∞ control problems can be found in [1, 3], [8, 9].

2. Theoretical Background

In this section the disturbance rejection problem is considered on the assumption that the full state vector can be measured. If this is not the case, one may always use an observer to reconstruct the state variables, as it is usual in structural control applications, and then one may continue by using the model presented here [5]. Let us consider the system

$$\dot{x}(t) = Ax(t) + Bz(t) + Dq(t), \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where $x(t) \in \Re^n$, $z(t) \in \Re^m$, $q(t) \in \Re^g$, $y(t) \in \Re^p$ and A, B, C, D are constant matrices of appropriate dimensions. The vector $q(t)$ denotes the unmeasurable disturbance. In this paper this is the earthquake excitation.

In the frequency domain the above system takes the form:

$$sX(s) = AX(s) + BU(s) + DZ(s), Y(s) = CX(s). \tag{3}$$

Let the system be left invertible, i.e. $rank\left[C(sI - A)^{-1}B\right] = m, \forall s \in C$.

The disturbance rejection problem is stated as follows: Find matrices F and G such that when a feedback control law of the form

$$z = Fx + G\omega \tag{4}$$

is applied, where $\omega(t) \in \Re^m$ is the new input vector and G is assummed to be invertible, the following relation holds:

$$\left[\; C\,(sI - A - BF)^{-1}\,BG \;\; C\,(sI - A - BF)^{-1}\,D \;\right] = \left[\; H(s) \;\; O_{p\times g} \;\right]. \tag{5}$$

The latter relation means that the influence of the disturbances on the system output is eliminated, since:

$$Y(s) = C\,(sI - A - BF)^{-1}\,BG\Omega(s) + C\,(sI - A - BF)^{-1}\,BDQ(s). \tag{6}$$

In the frequency domain the relation (4) takes on the form:

$$Z(s) = FX(s) + G\Omega(s), \Omega(s) \in C^m. \tag{7}$$

Let $W(s) = \prod^{i=1,\alpha} S_{\alpha-1}M_{\alpha-1}$ is applied on the system, where M_i are invertible matrices which are defined as follows:

$$M_0 = s^d I_p, M_{i+1} = \begin{bmatrix} I_{q_i} & 0 \\ 0 & sI_{p-q_i} \end{bmatrix},$$

$$d = \min_i \left\{CA^i \left[\; B \;\; D \;\right] \neq 0\right\}, i = 0, 1, \cdots, \tag{8}$$

and S_i are invertible matrices such that:

$$S_i C_i \begin{bmatrix} B & D \end{bmatrix} = \begin{bmatrix} C_i^+ \\ \widetilde{C}_i \end{bmatrix} \begin{bmatrix} B & D \end{bmatrix} = \begin{bmatrix} C_i^+ \\ 0 \end{bmatrix} \begin{bmatrix} B & D \end{bmatrix}, i = 0, 1, \cdots, \tag{9}$$

$$C_0 = CA^d, C_i = \begin{bmatrix} C_{i-1}^+ \\ \widetilde{C}_{i-1} A \end{bmatrix}, i = 0, 1, \cdots$$

Here $q_i = rank \left\{ C_i \begin{bmatrix} B & D \end{bmatrix} \right\}$, $C_i^+ \begin{bmatrix} B & D \end{bmatrix}$ is a $q_i \times m$ full row rank matrix and α is the smallest integer for which $q_\alpha = m$.

From the construction of the matrices S_i, M_i and C_i and by using the Cayley-Hamilton theorem, one may readily show that $\alpha \leqslant n - d - 1$, as well as that:

$$W(s)C(sI - A)^{-1} \begin{bmatrix} B & D \end{bmatrix} = \begin{bmatrix} \widehat{C} \\ 0 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} B & D \end{bmatrix}, \widehat{C} = C_\alpha^+. \tag{10}$$

The necessary and sufficient conditions for the above described disturbance rejection problem to have a solution are [3]:

i.

$$rank \left[C(sI - A)^{-1} \begin{bmatrix} B & D \end{bmatrix} \right] = m, \text{ and} \tag{11}$$

ii.

$$\widehat{C} D = 0. \tag{12}$$

When these conditions are satisfied, matrix G may be any arbitrary invertible matrix and a special solution F^* exists for F, which is given by the relation: $F^* = -\left(\widehat{C}B\right)^{-1} \widehat{C}A$.

Let $A_C = A + BF^* = A - B\left(\widehat{C}B\right)^{-1} \widehat{C}A$,
$L = \begin{bmatrix} D & A_C D & \cdots & A_C^{n-1} D \end{bmatrix}$, $\lambda = rank\ L$, and N be the $(n - \lambda) \times n$ full row rank matrix which is orthogonal to the matrix L. The general analytical expression of the disturbance rejection matrix F is: $F = -\left(\widehat{C}B\right)^{-1} \widehat{C}A + TN$, where the only free parameters (used for economy, stability and other design requirements) are the elements of the $m \times (n - \lambda)$ arbitrary matrix T.

3. Mathematical Model of the Civil Structure

The dynamical model of a linearly elastic structure with a m-dimensional control vector z reads:

$$M\ddot{u} + \check{c}\dot{u} + Ku = M_0 \ddot{u}_g + B_0 z(t), \tag{13}$$

where M is the $k \times k$ mass matrix, $\check{c}$ is the $k \times k$ damping matrix, K is the $k \times k$ stiffness matrix, $\ddot{u}_g$ is g-dimensional ground earthquake acceleration vector, $\ddot{u}$ is the k-dimensional acceleration vector, $\dot{u}$ is the k-dimensional velocity vector and u is the k-dimensional nodal displacement vector. Moreover, B_0 is the $k \times m$ control forces arrangement matrix. Here, the method of additional masses is used for the approximate modelling of the structure with the ground support earthquake acceleration $\ddot{u}_g$ (see, e.g., [2], page 76). The additional mass of the ground is used for the construction of matrices M and M_0 in (13). By using the substitution: $\dot{u} = v$, one gets from (13) the state space model:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}\check{c} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}M_0 \end{bmatrix} \ddot{u}_g + \begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix} z(t). \tag{14}$$

Equivalently, these relations can be written as:

$$\dot{x} = Ax + Bz + Dq,\ y(t) = Cx(t), \tag{15}$$

where A, B, C, D are certain $n \times n$, $n \times m$, $p \times n$, $n \times g$ $(n = 2k)$ matrices respectively, under the following substitutions:

$$A = \begin{bmatrix} 0_{k\times k} & I_{k\times k} \\ -M^{-1}K & -M^{-1}\check{c} \end{bmatrix},\ D = \begin{bmatrix} 0_{k\times g} \\ M^{-1}M_0 \end{bmatrix},\ B = \begin{bmatrix} 0_{k\times m} \\ M^{-1}B_0 \end{bmatrix}, \tag{16}$$

$$x = \begin{bmatrix} \dot{u} \\ v \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{u} \end{bmatrix},\ \text{and } q(t) = \ddot{u}_g. \tag{17}$$

The above general formulation for active control system can be specified for the structure as follows, by accepting a diagonal structure of mass matrix M. To this end one assumes as output the displacements of the nodes at the story over the foundation where the control force inputs are placed (control floor), i.e.

$$C = \begin{bmatrix} 0_{m\times g} & I_{m\times m} & 0_{m\times r} & 0_{m\times g} & 0_{m\times m} & 0_{m\times r} \end{bmatrix}. \tag{18}$$

Let the following partition of displacement, velocity and acceleration degrees of freedom be considered:

$$u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T,\ \dot{u} = \begin{bmatrix} \dot{u}_1 & \dot{u}_2 & \dot{u}_3 \end{bmatrix}^T \text{ and } \ddot{u} = \begin{bmatrix} \ddot{u}_1 & \ddot{u}_2 & \ddot{u}_3 \end{bmatrix}^T,$$

where u_1 is the g-dimensional vector of basement displacements, u_2 is the m-dimensional vector of control floor displacements and u_3 is the r-displacement vector of the rest (upper stories) displacements. Since a localized ground story control configuration is studied in this paper, a partitioned form of (14) is more relevant to our investigation. It reads:

$$\begin{bmatrix} M_1+M_g & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} \check{c}_{11} & \check{c}_{12} & \check{c}_{13} \\ \check{c}_{21} & \check{c}_{22} & \check{c}_{23} \\ \check{c}_{31} & \check{c}_{32} & \check{c}_{33} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ B_m \\ 0 \end{bmatrix} z + \begin{bmatrix} M_g \\ 0 \\ 0 \end{bmatrix} \ddot{u}_g, \tag{19}$$

since $B_0 = \begin{bmatrix} 0_{g\times m} \\ B_{m\times m} \\ 0_{r\times m} \end{bmatrix}$, $\Gamma = \begin{bmatrix} M_g \\ 0_{m\times g} \\ 0_{r\times g} \end{bmatrix}$, where B_m is an $m \times m$ invertible matrix to ensure the linear independence of the control imputs z and M_g is an appropriate fictitious additional mass matrix of dimension $g \times l$. This is a general formulation, which includes for example rocking motions, where one has l elements of measured ground accelerations and g degrees of freedom at the ground nodes. In the following we set $\overline{M}_1 = M_1 + M_g$. Thus, equations (16) take on the form:

$$A = \begin{bmatrix} 0_{k\times k} & I_{k\times k} \\ -A_1 & -A_2 \end{bmatrix} \tag{20}$$

$$A_1 = \begin{bmatrix} \overline{M}_1^{-1}K_{11} & \overline{M}_1^{-1}K_{12} & \overline{M}_1^{-1}K_{13} \\ M_2^{-1}K_{21} & M_2^{-1}K_{22} & M_2^{-1}K_{23} \\ M_3^{-1}K_{31} & M_3^{-1}K_{32} & M_3^{-1}K_{33} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \overline{M}_1^{-1}\check{c}_{11} & \overline{M}_1^{-1}\check{c}_{12} & \overline{M}_1^{-1}\check{c}_{13} \\ M_2^{-1}\check{c}_{21} & M_2^{-1}\check{c}_{22} & M_2^{-1}\check{c}_{23} \\ M_3^{-1}\check{c}_{31} & M_3^{-1}\check{c}_{32} & M_3^{-1}\check{c}_{33} \end{bmatrix},$$

$$B = \begin{bmatrix} 0_{k\times m} \\ M^{-1}B_0 \end{bmatrix} = \begin{bmatrix} 0_{k\times m} \\ 0_{g\times m} \\ M_2^{-1}B_m \\ 0_{r\times m} \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0_{k\times g} \\ M^{-1}\Gamma \end{bmatrix} = \begin{bmatrix} 0_{k\times g} \\ \overline{M}_1^{-1}M_g \\ 0_{m\times g} \\ 0_{r\times g} \end{bmatrix}.$$

Note that in (13), x, z are supposed to be dependent on time, while all other quantities (i.e., A, B, C, D) are assumed to be time invariant, as usual in linear elastodynamic analysis.

4. Seismic Disturbance Rejection of Civil Structures

The general theory outlined in section two is applied on the specific model of section three. In the case that we use the partitioned form of the structure described by the equations (19) it will be shown that the problem of exact disturbance rejection is solvable for the above civil engineering structure. The examined system is left invertible since:

$$CB = 0, \ \text{and} \tag{21}$$

$$CAB = M^{-1}B_0 = M_2^{-1}B_m, \tag{22}$$

i.e. CAB is an invertible matrix. For left invertible systems it is proved [3] that the necessary and sufficient condition for the problem of disturbance rejection to have a solution are the relations (11), (12). Relation (11) always holds true in the present case since relation (22) holds and rankCAB = rank$M_2^{-1}B_m = m$.

In order to check condition (12) one may use the algorithm proposed in [4] and exploit the structure form of the control system introduced in the previous section. For this system one has $d = 1$, since the relations (21), (22) hold and CAB is an invertible matrix. Thus, one stops at the first step of the above algorithm [3] and consequently one has: $\widehat{C} = C_0 = CA$. Thus, to satisfy condition (11) it must be: $CAD = 0$. But the latter relation always holds true in the control configuration considered here. Thus for the localized ground story control configuration of (18) the disturbance rejection problem is always solvable if the control floor displacements are assumed to be the output of the system (see (17)).

So, exact disturbance rejection can be achieved for the controlled structure configuration described previously. Since the required conditions are satisfied, G may be any arbitrary invertible matrix and a special solution F^* exists for F, which is given by:

$$F^* = -\left(\widehat{C}B\right)^{-1}\widehat{C}A = B_m^{-1}\begin{bmatrix} K_{21} & K_{22} & K_{23} & \check{c}_{21} & \check{c}_{22} & \check{c}_{23} \end{bmatrix}. \tag{23}$$

Let $A_C = \begin{bmatrix} 0_{k\times k} & I_{k\times k} \\ -A_3 & -A_4 \end{bmatrix}$, with $A_3 = \begin{bmatrix} \overline{M}_1^{-1}K_{11} & \overline{M}_1^{-1}K_{12} & \overline{M}_1^{-1}K_{13} \\ 0_{m\times g} & 0_{m\times m} & 0_{m\times n_3} \\ M_3^{-1}K_{31} & M_3^{-1}K_{32} & M_3^{-1}K_{33} \end{bmatrix}$,

$$A_4 = \begin{bmatrix} \overline{M}_1^{-1}\check{c}_{11} & \overline{M}_1^{-1}\check{c}_{12} & \overline{M}_1^{-1}\check{c}_{13} \\ 0_{m\times g} & 0_{m\times m} & 0_{m\times n_3} \\ M_3^{-1}\check{c}_{31} & M_3^{-1}\check{c}_{32} & M_3^{-1}\check{c}_{33} \end{bmatrix}, L = \begin{bmatrix} D & A_C D & \cdots & A_C^{n-1}D \end{bmatrix},$$

$\lambda = rank\ L$ and N the $(n-\lambda)\times n$ full row rank matrix which is orthogonal to the matrix L. The general analytical expression of the disturbance rejection matrix F is [6]: $F = F^* + TN$, where the only free parameters (used for economy, stability etc.) are the elements of the $m \times (n-\lambda)$ arbitrary matrix T. It is plausible that for controllable systems the solution for the matrix F is F^*, since $N = 0$.

5. Example: Shear-type Frame

The above general results have an application in shear-type frames. According to the shear-type frame concept, horizontal beams are supposed to be rigid. Thus external action on one story is concentrated in two consecutive stories and a control at the ground story can produce a rigid body motion at the upper stories. So, by eliminating displacements of the control floor (output), one achieves the elimination of the displacements of all upper stories as well. The two-story, single-bay steel frame of Fig. 1 is considered. All data are in compatible units in this academic example.

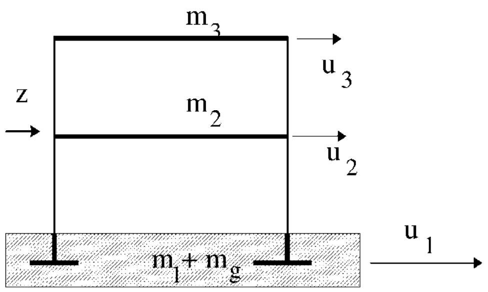

Figure 1.

We assume that the masses are $m_1 = m_2 = m_3 = 16$, the additional mass is set equal to $m_g = 5000 (\cong 100 \times \sum_{i=1,3} m_i)$ and that only one control force is applied, i.e., $m = 1$ and $B_0 = [010]^T$. Moreover, in this problem one has $k = 3$ degrees of freedom and one horizontal base acceleration ($g = 1$). The stiffness matrix is set equal to

$$K = \begin{bmatrix} 1500 & -1500 & 0 \\ -1500 & 3000 & 0 \\ 0 & -1500 & 1500 \end{bmatrix},$$

under the assumption of equal storeys with equivalent shear stiffness equal to 1500 for each story. A simple damping $\check{c} = diag\{1.6\}$ is assumed. Finally, the displacement of the first story is measured, i.e., $y = u_2$ and $C = [010000]$.

We these data one has, for example: $x = \begin{bmatrix} u_1 & u_2 & u_3 & \dot{u}_1 & \dot{u}_2 & \dot{u}_3 \end{bmatrix}^T$ and

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.2990 & 0.2990 & 0 & -0.0003 & 0 & 0 \\ 93.7500 & -187.5000 & 93.7500 & 0 & -0.1000 & 0 \\ 0 & 93.7500 & -93.7500 & 0 & 0 & -0.1000 \end{bmatrix}.$$

Application of the method presented in this paper yields the control law $z = Fx$, with F^* given by (23):

$$F^* = 1000.0 \times \begin{bmatrix} -1.5000 & 3.0000 & 1.5000 & 0 & 0.0016 & 0 \end{bmatrix}.$$

$$L = \begin{bmatrix} 0 & 312.50 & -0.0937 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 312.50 & -0.0937 & -93.4375 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } N = [0.2990 \quad 0,000299 \quad 1].$$

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